

EFFECTIVE EQUIDISTRIBUTION FOR SOME UNIPOTENT FLOWS IN $\mathrm{PSL}(2, \mathbb{R})^k$ MOD COCOMPACT, IRREDUCIBLE LATTICE

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ABSTRACT. Let $k \geq 2$, and let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})^k$ be an irreducible, cocompact lattice. We prove effective equidistribution for coordinate horocycle flows on $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})^k$. This is the simplest case for proving effective equidistribution of unipotent flows in this setting.

The main ingredients are Flaminio-Forni's analysis of the equidistribution of horocycle flows and a result by Kelmer-Sarnak on the strong spectral gap property of Γ in $\mathrm{PSL}(2, \mathbb{R})^k$.

1. INTRODUCTION

There has been greater interest recently in making Ratner's equidistribution theorems effective (see [10] and [11]). Green-Tao proved all Diophantine nilflows on any nilmanifold become equidistributed at polynomial speed, see [6]. Flaminio-Forni proved rather sharp estimates on the speed of equidistribution for a class of higher step nilmanifolds. Einsiedler-Margulis-Venkatesh proved effective equidistribution for large closed orbits of semisimple groups on homogeneous spaces, under some technical restrictions, in [2].

For any $k \geq 2$, we will prove effective equidistribution for coordinate horocycle flows on the quotient manifold given by the direct product of k copies of $\mathrm{PSL}(2, \mathbb{R})$ by a cocompact, irreducible lattice. Such flows are one-dimensional and non-horospherical unipotent flows on a (non-solvable) homogeneous space. To the author's knowledge, the first proof of effective equidistribution for unipotent subgroups which are not the full horospherical subgroup of a hyperbolic subgroup is due to Venkatesh in 2006, see [14]. He proved effective equidistribution for the product flow given by the horocycle flow and a circle translation on compact $\Gamma \backslash \mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}/\mathbb{Z}$. Very recently, the author and Vishe refined his approach and established a sharper rate of equidistribution that is independent of the spectral gap of the Laplacian, see [13]. At the same time, Flaminio, Forni and the author obtained a still sharper estimate (also independent of the spectral gap) for the equidistribution of this flow via a completely different method, see [5]. New results of Strömbergsson and Browning-Vinogradov established a rate of equidistribution for one-dimensional unipotent flows on $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \backslash \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, see [12] and [1].

We will now discuss in detail the setting for our equidistribution result. Let $k \geq 2$, and let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})^k$ be an irreducible, cocompact lattice. Let $M := \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})^k$. The vector fields on M are elements of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})^k$,

the direct product of k copies of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. For each $1 \leq i \leq k$, define the vector fields $\{U_i, V_i\}$ in $\mathfrak{sl}(2, \mathbb{R})^k$ by

$$\begin{aligned} U_i &:= (0, \dots, 0, U, 0, \dots, 0), \\ V_i &:= (0, \dots, 0, V, 0, \dots, 0), \end{aligned}$$

where U and V are elements of $\mathfrak{sl}(2, \mathbb{R})$ that occur in the i_{th} position of the k -tuple, and they are given by

$$U := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } V := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The set $\bigcup_{i=1}^k \{U_i, V_i\}$ is the complete set of generators for coordinate horocycle flows on M .

The space of diagonal elements in $\mathfrak{sl}(2, \mathbb{R})^k$ is k -dimensional. For each $1 \leq i \leq k$, define the diagonal element

$$X_i := (0, \dots, 0, X, 0, \dots, 0) \in \mathfrak{sl}(2, \mathbb{R})^k,$$

where $X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ occurs in the i_{th} position and generates the geodesic flow on $\text{PSL}(2, \mathbb{R})$.

For $i \in \{1, 2, \dots, k\}$. We will study the rate of equidistribution of the coordinate horocycle flow $\{\phi_t^i\}_{t \in \mathbb{R}}$ on M given by

$$\phi_t^i(x) = x e^{tU_i}.$$

It is unclear to the author how to proceed via effective decay of correlations of an auxiliary flow on M . In particular, effective equidistribution of coordinate horocycle flows cannot be easily derived from decay of correlations of a hyperbolic action on M .

Instead, we prove that essentially sharp (or sharp) estimates follow relatively easily from known works by way of unitary representations and invariant distributions. The main point is that the invariant distributions for $\{\phi_t^i\}_{t \in \mathbb{R}}$ are defined in unitary Sobolev representations of $\text{PSL}(2, \mathbb{R})$, so they are already well-understood by the work of Flaminio and Forni on the equidistribution of horocycle flows on $\text{PSL}(2, \mathbb{R}) \bmod \text{lattice}$ (see Theorem 1.1 and Theorem 1.4 of [3]). In fact, there is a basis of invariant distributions for the horocycle flow in [3] which are generalized eigendistributions for the geodesic flow. Working in one irreducible, unitary representation at a time, Sobolev estimates for both the scaling of these distributions under action of the geodesic flow and the solution of the cohomological equation of the horocycle flow only depend on the Sobolev order of the norm and that representation's Casimir parameter. By a recent result of Kelmer-Sarnak on the strong spectral gap property of irreducible, cocompact lattices in $\text{PSL}(2, \mathbb{R})^k$, we get that the positive spectrum of the Casimir parameter of the relevant $\text{PSL}(2, \mathbb{R})$ unitary representation of $L^2(M)$ is bounded away from zero. This means the results of [3] can be used, and more than that, the arguments there go through essentially without modification.

The success of using unitary representations here provides some evidence that they may also be useful in studying the related and more difficult problem of effective equidistribution of joinings of the horocycle flow on $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}) \bmod$ lattice.

Let $L^2(M)$ be the separable Hilbert space of complex-valued square-integrable functions on M with respect to the Haar measure vol . Let $C^\infty(M)$ be the space of smooth functions on M and let $\mathcal{D}'(M) = (C^\infty(M))'$ be its distributional dual space.

Any element of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})^k$ acts on $\mathcal{D}'(M)$ via the right regular representation. For each i , the center of the enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})^k$ contains the second-order differential operator

$$\square_i := [-X_i^2 - 1/2(U_i V_i + V_i U_i)] .$$

The Laplacian operator Δ is a second-order, elliptic element in the enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})^k$. For each $i \in \{1, \dots, k\}$, it is an essentially self-adjoint differential operator on $L^2(M)$ and is given by

$$\Delta := \Delta_i + \Delta_0 ,$$

where

$$\Delta_i := -X_i^2 - 1/2(U_i^2 + V_i^2) \text{ and } \Delta_0 := -\sum_{j \neq i} X_j^2 + 1/2(U_j^2 + V_j^2) .$$

The Sobolev space of order $s \in \mathbb{R}^+$ is the maximal domain $W^s(M)$ of the inner product

$$\langle f, g \rangle_s := \langle (I + \Delta)^s f, g \rangle ,$$

where I is the identity element of $\mathrm{PSL}(2, \mathbb{R})^k$. The space of s -order distributions on $W^s(M)$ is $W^{-s}(M) = (W^s(M))'$. Because M is compact, $C^\infty(M) = \bigcap_{s>0} W^s(M)$ and $\mathcal{D}'(M) = \bigcup_{s>0} W^{-s}(M)$.

Our estimates are given in terms of Sobolev norms involving a finite number of derivatives. In what follows, we let

$$s > 3k/2 + 1, \text{ and } i \in \{1, 2, \dots, k\} .$$

Let $\mathcal{W}_s(M)$ be the maximal domain of the operator $(I + \Delta_0)$ on $L^2(M)$ with inner product

$$\langle f, g \rangle_{\mathcal{W}_s(M)} := \langle (I + \Delta_0)^s f, g \rangle_{L^2(M)} .$$

Let $\pi : \mathrm{PSL}(2, \mathbb{R}) \rightarrow \mathcal{U}(\mathcal{W}_s(M))$ be a unitary representation of $\mathrm{PSL}(2, \mathbb{R})$ defined by

$$(1) \quad \pi(g)f(\Gamma(a, h, b)) = f(\Gamma(a, hg, b)) ,$$

for any $(a, h, b) \in \mathrm{PSL}(2, \mathbb{R})^{i-1} \times \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})^{k-i}$.

Now let $d\pi$ be the derived representation of π . The representation $d\pi$ is related to the derived representation of the right regular representation by the following simple lemma.

Lemma 1.1. *Let $Q \in \mathfrak{sl}(2, \mathbb{R})$, and let $Q_i := (0, \dots, 0, Q, 0, \dots, 0) \in \mathfrak{sl}(2, \mathbb{R})^k$, where Q is in the i_{th} -position. Then*

$$d\pi(Q) = Q_i.$$

Proof. Let $x \in M$ and $f \in C^\infty(M)$. For any $t \in \mathbb{R}$,

$$f(x(I^{i-1}, \exp(tQ), I^{k-i})) = f(x \exp(tQ_i)).$$

We conclude by differentiating at $t = 0$. □

Hence,

$$(2) \quad U_i = d\pi(U) \text{ and } (I + \Delta_i)f = d\pi(I - X^2 - 1/2(U^2 + V^2)).$$

Then with respect to a positive Stieltjes measure, $dm(\mu)$, the unitary representation π has the following direct integral decomposition

$$\mathcal{W}_s(M) = \int_{\oplus \mu \in \text{spec}(\square_i)} \mathcal{H}_{\mu,s} dm(\mu)$$

where the Casimir element \square_i acts as the constant μ on each unitary representation space $\mathcal{H}_{\mu,s}$. Each representation space $\mathcal{H}_{\mu,s}$ is a direct sum of an at most countable number of irreducible unitary representation spaces.

By irreducibility and (2), the vector fields U_i, X_i and V_i are *decomposable* into the irreducible representations of π in the sense that

$$(3) \quad W^s(M) = \int_{\oplus \mu \in \text{spec}(\square_i)} \mathcal{H}_\mu^s,$$

where $\mathcal{H}_\mu^s \subset \mathcal{H}_{\mu,s}$ inherits the inner product from $W^s(M)$.

The following lemma is a consequence of Theorem 2 of [7], as described in Section 1.3 of that paper.

Lemma 1.2 (Kelmer-Sarnak, [7]). *We have*

$$(4) \quad \inf \text{spec}(\square_i) \cap \mathbb{R}^+ > 0.$$

Proof. For any $j \in \mathbb{N} \setminus \{0\}$, and for a given infinite dimensional representation ρ of $\text{PSL}(2, \mathbb{R})^j$, let $p(\rho)$ be the infimum of all p such that there is a dense set of vectors v such that $\langle \rho(g)v, v \rangle$ is in $L^p(\text{PSL}(2, \mathbb{R})^j)$. The regular representation of $\text{PSL}(2, \mathbb{R})^k$ on $L^2(M)$ has a countable, orthogonal decomposition into irreducible, unitary representations ρ_m of $\text{PSL}(2, \mathbb{R})^k$. We may write $\rho_m = \rho_{m_1} \otimes \dots \otimes \rho_{m_k}$, where the ρ_{m_i} are irreducible, unitary representations of $\text{PSL}(2, \mathbb{R})$ in either the principal series, the complementary series or the discrete series.

We present the following special case of Theorem 2 of [7].

Theorem 1.1 (Kelmer-Sarnak). *Let $\Gamma \subset \text{PSL}(2, \mathbb{R})^k$ be an irreducible, co-compact lattice, and let ρ_m be as above. Then for any $\epsilon > 0$, $p(\rho_m) < 6 + \epsilon$, except for a finite number of m 's.*

Principal series and discrete series representations are tempered, that is, if ρ_{m_j} is such a representation, then for all j , $p(\rho_{m_j}) = 2$. Complementary series representations can be parameterized by $\nu_j \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, where $p(\rho_{m_j}) = \max\{1/\nu_j, 1/(1 -$

$\nu_j)\} \in \mathbb{R}^+$, which is unbounded if (4) is not true. We also have $p(\rho_m) = \max_j p(\rho_{m_j})$, so Theorem 1.1 shows that for an at most finite number of exceptions m ,

$$\max\{1/\nu_i, 1/(1 - \nu_i)\} \leq p(\rho_m) < 7.$$

This implies the lemma. \square

We denote the space of distributions in $W^{-s}(M)$ that are invariant under U_i by

$$\mathcal{J}^s(M) := \{D \in W^{-s}(M) : U_i D = 0\}.$$

Let $\mathcal{J}(M) := \bigcup_{s>0} \mathcal{J}^s(M)$. By (2), the classification of U_i -invariant distributions in $\mathcal{J}(M)$ into irreducible, unitary representation spaces is given by the corresponding classification of horocycle flow-invariant distributions from Theorem 1.1 of [3].

Let σ_{pp} be the eigenvalues of Δ_i on $L^2(M)$, which by the representation theory of $\mathrm{SL}(2, \mathbb{R})$, coincides with the positive eigenvalues of \square_i on $L^2(M)$.

Theorem 1.2 (Flaminio-Forni, [3]). *The space $\mathcal{J}(M)$ has infinite countable dimension. There is a decomposition*

$$\mathcal{J}(M) = \bigoplus_{\mu \in \sigma_{pp}} \mathcal{J}_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} \mathcal{J}_n \oplus \mathcal{J}_c,$$

where

- for $\mu = 0$, the space \mathcal{J}_0 is spanned by the Haar measure on M ;
- for $0 < \mu < 1/4$, there is a splitting $\mathcal{J}_\mu = \mathcal{J}_\mu^+ \oplus \mathcal{J}_\mu^-$, where $\mathcal{J}_\mu^\pm \subset W^{-s}(M)$ if and only if $s > \frac{1 \pm \sqrt{1-4\mu}}{2}$, and each subspace has dimension equal to the multiplicity of $\mu \in \sigma_{pp}$;
- for $\mu \geq \frac{1}{4}$, the space $\mathcal{J}_\mu \subset W^{-s}(M)$ if and only if $s > 1/2$, and it has dimension equal to twice the multiplicity of $\mu \in \sigma_{pp}$;
- for $n \in \mathbb{Z}_{\geq 2}$, the space $\mathcal{J}_n \subset W^{-s}(M)$ if and only if $s > n/2$ and it has dimension equal to twice the multiplicity of $\mu = \frac{1}{4}(-n^2 + 2n) \in \mathrm{spec}(\square_i)$;
- the space $\mathcal{J}_c \subset W^{-s}(M)$ if and only if $s > 1/2$. It is defined on the continuous spectrum of Δ_i on $W^{-s}(M)$, and it has infinite countable dimension.

For $s > 1/2$, Theorem 1.4 of [3] shows $\mathcal{J}(M)$ has a countable basis \mathcal{B}^s of unit normed (in $W^{-s}(M)$), generalized eigenvectors for the geodesic flow $\{e^{tX_i}\}_{t \in \mathbb{R}}$.

For any $s > 1$, let

$$\mathcal{B}_+^s := \bigcup_{\mu \in \sigma_{pp}/\{\frac{1}{4}\}} \mathcal{B}^s \cap \mathcal{J}_\mu^s,$$

be a basis of U_i -invariant distributions for $\left(\bigoplus_{\mu \in \sigma_{pp}/\{\frac{1}{4}\}} \mathcal{J}_\mu\right)$. Let

$$\mathcal{B}_-^s := \left(\bigcup_{\mu \geq \frac{1}{4}} \mathcal{B}^s \cap \mathcal{J}_\mu^s \right) / \mathcal{B}_+^s.$$

be a basis of invariant distributions for the rest of principal series. It will also be convenient to define

$$\mathcal{B}_{1/4}^s := \mathcal{B}^s \cap \mathcal{J}_{1/4}^s.$$

For $D \in W^{-s}(M)$, let

$$\mathcal{S}_D := \begin{cases} \frac{1 \pm \operatorname{Re} \sqrt{1-4\mu}}{2} & \text{if } D \in \mathcal{J}_\mu^\pm, \mu > 0; \\ n/2 & \text{if } D \in \mathcal{J}_n, n \in \mathbb{Z}_{\geq 2}; \\ 1/2 & \text{if } D \in \mathcal{J}_c. \end{cases}$$

We remark that by Lemma 1.2,

$$\inf \left\{ \frac{1 - \operatorname{Re} \sqrt{1-4\mu}}{2} > 0 : \mu \in \operatorname{spec}(\square_i) \cap \mathbb{R}^+ \right\} > 0.$$

Now for $T \geq 1$, let $\log^+ T := \max\{1, \log T\}$.

Theorem 1.3. *Let $s > 3k/2 + 1$ and $k \geq 2$. Let $\Gamma \subset \operatorname{PSL}(2, \mathbb{R})^k$ be a cocompact, irreducible lattice.*

Then there is a constant $C_s := C_s(\Gamma) > 0$ such that for all $(x, T) \in M \times \mathbb{R}_{\geq 1}$, and there are real numbers $\{c_D(x, T)\}_{D \in \mathcal{B}_+^s \cup \mathcal{B}_-^s}$ and distributions $D_{x,T}^{s,2}, \mathcal{R}_{x,T}^s \in W^{-s}(M)$ such that the following estimate holds.

For all $f \in W^s(M)$,

$$\begin{aligned} \frac{1}{T} \int_0^T f \circ \phi_t^i(x) dt - \int_M f d\operatorname{vol} &= \sum_{D \in \mathcal{B}_+^s} c_D(x, T) D(f) T^{-\mathcal{S}_D} \\ &+ \sum_{D \in \mathcal{B}_-^s} c_D(x, T) D(f) T^{-1/2} \log^+ T \\ &+ \frac{D_{x,T}^{s,2}(f) \log^+ T + \mathcal{R}_{x,T}^s(f)}{T}, \end{aligned}$$

where for all $(x, T) \in M \times \mathbb{R}_{\geq 1}$,

$$\sum_{D \in \mathcal{B}_+^s \cup \mathcal{B}_-^s} |c_D(x, T)|^2 + \|D_{x,T}^{s,2}\|_{-s}^2 + \|\mathcal{R}_{x,T}^s\|_{-s}^2 \leq C_s.$$

Additionally, we have the following lower bound. For every $D \in \mathcal{B}^s$, there is a constant $C'_s := C'_s(D) > 0$ such that for all sufficiently large $T \gg 1$,

$$(5) \quad \|c_D(\cdot, T)\|_{L^2(M)} \geq \begin{cases} C'_s & \text{if } D \notin \mathcal{B}_{1/4}^s; \\ C'_s \log^+ T & \text{if } D \in \mathcal{B}_{1/4}^s. \end{cases}$$

Remark 1.1. *For sufficiently large $T \gg 1$, the upper bound for the above coefficients is sharp up to, possibly, multiplication by $\log(T)$.*

2. COHOMOLOGICAL EQUATION

For any $s > 0$, define

$$\text{Ann}^s(M) := \{f \in W^s(M) : D(f) = 0 \text{ for all } D \in \mathcal{I}^s(M)\}.$$

As a consequence of Theorem 1.2 of [3], we derive

Theorem 2.1. *Let $0 \leq r < s - 1$. Then for any $f \in \text{Ann}^s(M)$, there exists a function $g \in W^r(M)$, unique up to additive constants, such that*

$$U_i g = f.$$

Moreover, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ such that

$$\|g\|_r \leq C_{r,s} \|f\|_s.$$

Proof. First say r is an integer, and let s be as in the theorem. By (2), by Theorem 1.2 of [3] and by Theorem 2 and Section 1.3 of [7], we have that for any $f \in \text{Ann}^s(M)$ there is a function $g \in W^r(M)$ and a constant $C_{r,s,\Gamma} > 0$ such that

$$U_i g = f$$

and

$$(6) \quad \|(I + \Delta_i)^r g\|_{L^2(M)} \leq C_{r,s,\Gamma} \|(I + \Delta_i)^s f\|_{L^2(M)}.$$

Now fix f and g as in the theorem. For $\{c_n\}_{n=0}^r \subset \mathbb{Z}^+$, write

$$(I + \Delta)^r = \sum_{n=0}^r c_n \Delta^n.$$

and observe that

$$(7) \quad \Delta^n = \sum_{m=0}^n \Delta_i^m \Delta_0^{n-m}.$$

Because U_i commutes with Δ_0 , we have

$$U_i \Delta_0 g = \Delta_0 f.$$

Then for any $\epsilon > 0$, (6) gives a constant $C_{\epsilon,\Gamma} > 0$ such that

$$(8) \quad \|(I - \Delta_i)^m \Delta_0^{n-m} g\|_{L^2(M)} \leq C_{\epsilon,\Gamma} \|(I - \Delta_i)^{m+1+\epsilon} \Delta_0^{n-m} f\|_{L^2(M)},$$

By Lemma 6.3 of [9], for all $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, there is a constant $C_{\alpha+\beta} > 0$ such that

$$(I - \Delta_i)^\alpha \Delta_0^\beta \leq C_{\alpha+\beta} (I + \Delta)^{\alpha+\beta}.$$

By interpolation, the same holds for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$, see [8]. Hence,

$$(9) \quad (I - \Delta_i)^{2(m+1+\epsilon)} \Delta_0^{2(n-m)} \leq C_\epsilon (I + \Delta)^{2n+1+\epsilon}.$$

Letting $\epsilon < s - 2n - 1$, and combining (7), (8) and (9), we have

$$\langle (1 + \Delta)^{2r} g, g \rangle \leq C_{r,s,\Gamma} \langle (1 + \Delta)^{2s} f, f \rangle.$$

This finishes the proof of Theorem 2.1 in the case when $r \in \mathbb{Z}_{\geq 0}$. The general case for $r \geq 0$ follows by interpolation. \square

3. PROOF OF THEOREM 1.3

For all $(x, T) \in M \times \mathbb{R}^+$, write $\gamma_{x,T}$ as

$$\gamma_{x,T}(f) := \frac{1}{T} \int_0^T f \circ \phi_t^i(x) dt.$$

We may orthogonally project $\gamma_{x,T}$ in $W^{-s}(M)$ onto a basis of $\{\phi_t^i\}_t$ -invariant distributions described in Theorem 1.2. Let $\mathcal{C}_{\gamma_{x,T}}$ be orthogonal projection in $W^{-s}(M)$ of $\gamma_{x,T}$ into $\langle \bigcup_{D \in \mathcal{B}_-^s / \mathcal{B}_{1/4}^s} D \rangle$. For each $D \in \mathcal{B}^s \cap \langle \mathcal{C}_{\gamma_{x,T}} \rangle^\perp$, let $D_{\gamma_{x,T}}$ be orthogonal projection in $W^{-s}(M)$ of $\gamma_{x,T}$ onto $\langle D \rangle$. Then there is a remainder $R_{\gamma_{x,T}}$ such that

$$(10) \quad \gamma_{x,T} = \left(\sum_{D \in \mathcal{B}^s \cap \langle \mathcal{C}_{\gamma_{x,T}} \rangle^\perp} D_{\gamma_{x,T}} \right) \oplus \mathcal{C}_{\gamma_{x,T}} \oplus R_{\gamma_{x,T}}.$$

Notice that the space of invariant distributions in each irreducible, unitary representation is at most two dimensional. Lemma 5.2 of [3] gives that for some constant $C_s := C_s(\Gamma) > 0$, the quantity

$$(11) \quad \sum_{D \in \mathcal{B}^s(M)} \|D_{\gamma_{x,T}}\|_{-s}^2 + \|\mathcal{C}_{\gamma_{x,T}}\|_{-s}^2 + \|R_{\gamma_{x,T}}\|_{-s}^2.$$

satisfies

$$(12) \quad C_s^{-2} \|\gamma_{x,T}\|_{-s}^2 \leq (11) \leq C_s^2 \|\gamma_{x,T}\|_{-s}^2.$$

We prove Theorem 1.3 by estimating each of the terms in (11).

Lemma 3.1. *Let $s > \frac{3k}{2} + 1$. There is a constant $C_s := C_s(\Gamma) > 0$ such that for any $x \in M$ and any $T > 0$,*

$$\|R_{\gamma_{x,T}}\|_{-s} \leq \frac{C_s}{T}.$$

Proof. Let $f \in \text{Ann}^s(M)$. Then by Theorem 2.1, for any $\frac{3k}{2} < r < s - 1$, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ and a function $g \in W^s(M)$ satisfying $U_i g = f$ and

$$\|g\|_r \leq C_{r,s} \|f\|_s.$$

Then as in Lemma 5.5 of [3], we get by the Sobolev embedding theorem that

$$\begin{aligned} |R_{\gamma_{x,T}}(f)| &= \frac{1}{T} \left| \int_0^T f \circ \phi_t^i(x) dt \right| \\ &= \frac{1}{T} \left| \int_0^T U_i g \circ \phi_t(x) dt \right| \\ &= \frac{|g \circ \phi_T^i(x) - g(x)|}{T} \\ &\leq \frac{C_r}{T} \|g\|_r \leq \frac{C_{r,s}}{T} \|f\|_s. \end{aligned}$$

The dependence of $C_{r,s}$ on r can be removed by taking $r = s/2 + (3k - 2)/4$. \square

Lemma 3.2. *For every $s > \frac{3k}{2} + 1$, there is a constant $C_{s,\Gamma} > 0$ such that the following holds. For any $\mu \in \sigma_{pp}/\{\frac{1}{4}\}$, for any $D \in \mathcal{J}_\mu^\pm \cap \mathcal{B}^s$, and for any $x \in M$ and $T > 1$, the distribution $D_{\gamma_{x,T}}$ from (10) satisfies*

$$\|D_{\gamma_{x,T}}\|_{-s} \leq C_{s,\Gamma} T^{-s_D}.$$

Proof. Using Lemma 1.1, the argument is the same as in Section 5.3 of [3]. We give it here for the convenience of the reader.

For any $x \in M$, for any $T \geq 1$ and for any $t \in \mathbb{R}$, we have

$$(13) \quad e^{tX_i} \gamma_{x,T} = \gamma_{xe^{-tX_i}, Te^t}.$$

Then fix x and T as in the lemma, and note $e^{-\log T X_i} \gamma_{x,T} = \gamma_{xe^{\log T X_i}, 1}$. By Theorem 1.2, $\mathcal{D}_{x,T}$ is a generalized eigendistribution for the geodesic flow. Because the orthogonal splitting $\mathcal{J}^s(M) \oplus \mathcal{J}^s(M)^\perp$ is not preserved under $\{e^{tX_i}\}_t$, we do not immediately get an estimate of $\|\mathcal{D}_{x,T}\|_{-s}$. Instead, we estimate it by an iterative argument.

Let $h \in [1, 2]$ be such that $e^{h\lfloor \log^+ T \rfloor} = T$. Using (13), for any $l \in \{0, \dots, \lfloor \log^+ T \rfloor - 1\}$, we have

$$(14) \quad \begin{aligned} \|D_{\gamma_{xe^{(\log T - (l+1)h)X_i}, e^{(l+1)h}}}\|_{-s} &\leq \|\exp(hX_i) \mathcal{D}_{\gamma_{xe^{(\log T - lh)X_i}, e^{lh}}}\|_{-s} \\ &\quad + \left\| \left(\exp(hX_i) \mathcal{R}_{xe^{(\log T - lh)X_i}, e^{lh}} \right) \right\|_{-s} \end{aligned}$$

Now by Theorem 1.4 of [3] and Lemma 1.1, D is an eigendistribution of e^{hX_i} with eigenvalue $e^{-h(1 \pm \sqrt{1-4\mu})/2} = e^{-h s_D}$. Hence, the same is true for $\mathcal{D}_{\gamma_{xe^{(\log T - lh)X_i}, e^{lh}}}$, for any l . Also, e^{hX_i} is a bounded operator. So there is a constant $C > 0$ such that

$$(15) \quad (14) \leq e^{-h s_D} \|\mathcal{D}_{\gamma_{xe^{(\log T - lh)X_i}, e^{lh}}}\|_{-s} + C \|\mathcal{R}_{xe^{(\log T - lh)X_i}, e^{lh}}\|_{-s}.$$

Recall that $(\gamma_{x,T})|_{\langle D \rangle} = D_{x,T}$, so we iterate and get

$$(16) \quad \begin{aligned} \|(\gamma_{x,T})|_{\langle D \rangle}\|_{-s} &\leq T^{-s_D} \|\mathcal{D}_{\gamma_{xe^{\log T X_i}, 1}}\|_{-s}^2 \\ &\quad + CT^{-s_D} \sum_{l=1}^{\lfloor \log^+ T \rfloor} e^{lh s_D} \|\mathcal{R}_{xe^{(\log^+ T - lh)X_i}, e^{lh}}\|_{-s}. \end{aligned}$$

Moreover, Lemma 3.1 guarantees that each $\|\mathcal{R}_{xe^{(\log^+ T - lh)X_i}, e^{lh}}\|_{-s}$ is small, so the above series converges. \square

Proof of Theorem 1.3. By Lemma 3.2, it remains to prove the upper bounds for distributions in $\mathcal{B}^s \cap (\mathcal{J}_c \cup \mathcal{J}_{1/4})$ and $\mathcal{B}^s \cap \mathcal{J}_n$, for $n \in \mathbb{N}_{\geq 2}$. By Lemma 5.1 of [3] and Lemma 1.1, there is a constant $C_s > 0$ such that for all $t \in \mathbb{R}$,

$$(17) \quad \|\exp(tX_i)D\|_{-s} \leq C_s(1 + |t|)e^{-t/2}.$$

Then by replacing $e^{-(1 \pm \operatorname{Re} \sqrt{1-4\mu})/2}$ with $(1 + |h|)e^{-1/2}$ in formula (15) of the above argument, we deduce that there is a constant $C_{s,\Gamma} > 0$ such that

$$\|D_{\gamma_{x,T}}\|_{-s} \leq C_{s,\Gamma} T^{-(1 \pm \operatorname{Re} \sqrt{1-4\mu})/2} \log^+ T.$$

For $D \in \mathcal{B}^s \cap \mathcal{J}_{1/4}$, Theorem 1.4 of [3] and Lemma 1.1 show that D is a generalized eigendistribution for e^{hX_i} satisfying (17) for all $t \in \mathbb{R}$.

For $D \in \mathcal{B}^s \cap \mathcal{J}_n$ and $n \in \mathbb{N}_{\geq 2}$, and for h as in Lemma 3.2, Theorem 1.4 of [3] gives that D is a eigendistribution for $\exp(hX_i)$ with eigenvalue $e^{-nh/2}$. Then the above argument gives, for any $x \in M$ and any $T \geq 1$,

$$(18) \quad \|D_{\gamma_{x,T}}\|_{-s} \leq C_{s,\Gamma} T^{-1} \log^+ T \quad \text{if } n = 2;$$

$$(19) \quad \|D_{\gamma_{x,T}}\|_{-s} \leq C_{s,\Gamma} T^{-1} \quad \text{if } n \in \mathbb{N}_{\geq 3}.$$

Now we define the remainder distribution $\mathcal{R}_{x,T}^s$ appearing in Theorem 1.3 as the orthogonal sum of the distribution $R_{\gamma_{x,T}}$ and the distributions $D_{\gamma_{x,T}}$ from (19). The estimate of $\|\mathcal{R}_{x,T}^s\|_{-s}$ follows from Lemma 3.1, formula (19) and orthogonality. Lastly, for $n = 2$, we define

$$D_{x,T}^{s,2} := D_{\gamma_{x,T}},$$

so (18) gives the estimate of $\|D_{x,T}^{s,2}\|_{-s}$. This concludes the proof for the upper bounds of the distributions in Theorem 1.3.

The L^2 lower bounds can be obtained by an argument involving the L^2 version of the Gottschalk-Hedlund Lemma. Using Lemma 1.1, the lower bound follows from Lemma 5.7, Lemma 5.8, Lemma 5.9 and Lemma 5.13 of [3]. We give that argument here for the convenience of the reader.

The Gottschalk-Hedlund Lemma says, in particular, that if f is not a coboundary for $\{\phi_t^i\}_t$, then the family of functions $\{T^{\gamma_{x,T}}(f)\}_{T \geq 1}$ on M is not equibounded in the L^2 -norm.

So let $D \in \mathcal{B}_{\pm}^s$. Then we can find a function $f \in W^s(M) \cap \operatorname{Ann}^s(M)^\perp$ such that $D(f) = 1$ and $\bar{D}(f) = 0$ for all $\bar{D} \in \mathcal{B}_{\pm}^s \setminus \langle D \rangle$. So for all $x \in M$ and $T \geq 1$,

$$\gamma_{x,T}(f) = c_D(x, T).$$

Hence,

$$(20) \quad \sup_{T \geq 1} T \|c_D(\cdot, T)\|_0 = +\infty.$$

If $D \notin \mathcal{B}_{1/4}^s$, then for any $m \in \mathbb{N}$,

$$\|c_D(\cdot, Te^m)\|_0 \geq e^{-m\mathcal{S}_D} \|c_D(\cdot, T)\|_0 - \mathcal{E}_D^s(x, T, m),$$

where $\mathcal{E}_D^s(x, T, m)$ is the contribution of the remainder given by

$$\begin{aligned} \mathcal{E}_D^s(x, T, m) &:= C_s e^{-m\mathcal{S}_D} \sum_{l=1}^m e^{l\mathcal{S}_D} \|\mathcal{R}_{xe^{(\log^+ T + (m-l)X_i)Te^l}}\|_{-s} \\ &\leq \frac{C_s}{T} e^{-m\mathcal{S}_D}, \end{aligned}$$

for some constant $C_s > 0$.

By (20), there is some $T > 1$ such that

$$\|c_D(\cdot, T)\|_0 \geq 2\frac{C_s}{T}.$$

This implies (5) in the case $D \notin \mathcal{B}_{1/4}^s$. The argument for $D \in \mathcal{B}_{1/4}^s$ is similar, see formulas (123) and (124) of [3] for details. \square

Proof of Remark 1.1. The proof is essentially given by Corollary 5.17 of [3]. Let μ_0 be the bottom of the positive spectrum of \square_i on $L^2(M)$. From the decomposition in Theorem 1.3 it is enough to consider the projection of $\gamma_{x,T}$ onto the invariant distributions supported in the part of the spectrum of \square_i containing μ_0 .

First suppose that $\mu_0 \in \sigma_{pp}$. Then it is enough to consider $D \in \mathcal{J}_{\mu_0}$, and let c_D be the coefficient for D in the decomposition from Theorem 1.3. Because pointwise upper bounds for c_D are the same order in T as the L^2 -lower bounds, we have a constant $C_s := C_s(\Gamma, D) > 0$ such that for any $x \in M$ and any $T \gg 1$ sufficiently large,

$$(21) \quad |c_D(x, T)| \leq C_s \|c_D(\cdot, T)\|_0.$$

Now we sketch the argument in Corollary 5.17 of [3], which proves the estimate in Theorem 1.3 is sharp.

Let $K \in (0, 1)$ be a constant, and let $A_T := A_{D,T,K} \subset M$ be defined by

$$A_T := \{x \in M : |c_D(x, T)| > K \|c_D(\cdot, T)\|_0\}.$$

Using (21), it follows that

$$(K^2 + (C_s)^2 \text{vol}(A_T)) \|c_D(\cdot, T)\|_0^2 \geq \|c_D(\cdot, T)\|_0^2.$$

It follows that there is a constant $\alpha := \alpha_{s,K,D} > 0$ such that for all $T \gg 1$ sufficiently large,

$$\text{vol}(A_T) \geq \alpha.$$

Now the L^2 lower bounds on c_D prove the remark in this case.

If μ_0 is contained in the continuous spectrum of \square_i , then $\mu_0 \geq \frac{1}{4}$, and by Theorem 1.2, the space \mathcal{J}_c is infinite dimensional. So there is a distribution $D \in \mathcal{B}_-^s \cap \mathcal{J}_c$ that is supported away from $\mu = \frac{1}{4}$. Because D is a direct integral of distributions with Casimir parameter $\mu > \frac{1}{4}$, Theorem 1.4 of [3] and Lemma 1.1 give

$$\|\exp(tX_i) \exp(\mathcal{D}_{\gamma_{xe^{\log TX_i,1}}})\|_{-s} = e^{-t/2} \|\mathcal{D}_{\gamma_{xe^{\log TX_i,1}}}\|_{-s}.$$

Then, as in Lemma 3.2, we get that $D_{\gamma_{x,T}}$ satisfies the following sharper estimate than in Theorem 1.3. There is a constant $C_{s,\Gamma} > 0$ such that for any $x \in M$ and any $T \geq 1$,

$$\|D_{\gamma_{x,T}}\|_{-s} \leq C_{s,\Gamma} T^{-1/2},$$

Hence, the pointwise upper bound on the coefficient c_D from Theorem 1.3 is the same order in T as its L^2 lower bound. The above argument implies that the pointwise upper bound for $|c_D(x, T)|$ is sharp on a set A_T of positive measure.

Comparing this bound with the upper bounds from all coefficients c_D in Theorem 1.3 proves Remark 1.1. \square

4. ACKNOWLEDGEMENTS

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